1.4: Korteweg-de Vries (KdV) equation

Here we consider the long-wave regime, where $k \to 0$, and assume that we can use the approximate dispersion relation

$$\omega = c_0 k - \beta k^3 \,, \tag{6}$$

with an error of $O(k^5)$. This translates to an evolution equation

$$u_t + c_0 u_x + \beta u_{xxx} = 0 \,, \tag{7}$$

where we recall that $-i\omega = \partial/\partial t, ik = \partial/\partial x$ for each Fourier component. The dominant term is $u_t + c_0 u_x \approx 0$, showing that the wave propagates with speed c_0 unchanged, except for the effect of the weak dispersion due to the term u_{xxx}. This small effect needs to be balanced by nonlinearity, and in many physical systems this has the form $\mu u u_x$, for some constant coefficient μ. Thus the model equation takes the form

$$u_t + c_0 u_x + \mu u u_x + \beta u_{xxx} = 0 \,. \tag{8}$$

This is the famous **Korteweg-de Vries (KdV)** equation, first derived in the water-wave context in 1895, and subsequently found to hold in many physical systems.